use of the knowledge of stability properties of the nonablating missile, to establish stability of the ablating missile. This approach gives simpler stability criteria than those obtained by directly applying Sinha's method to the equations of motion.

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Optimum Rectangular Radiative Fins Having Temperature-Variant Properties

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Nomenclature

a = parameter, Eq. (5)
b = fin thickness
B = beta function
$B_{(\,)}$ = incomplete beta function
k = thermal conductivity
k_0 = parameter, Eq. (4)
L = fin length
n = parameter, Eq. (4)
q = heat-transfer rate
T = temperature
x = coordinate
ε = emissivity
ε_0 = parameter, Eq. (5)
θ = transformed temperature
ξ = dimensionless coordinate
η = effectiveness
σ = Stefan-Boltzmann constant
ψ = dimensionless temperature
ψ_r = temperature ratio, θ_L/θ_b

Subscripts

b = fin base
L = fin tip
opt = optimum value
s = equivalent environment

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Introduction

THEORETICAL and experimental approaches to the problem of radiative heat transfer from fins possessing constant properties has been the subject of considerable study in recent years. However, a slightly more complicated state of affairs occurs when the properties are affected by temperature. As a result, only few investigations have appeared in the heat-transfer literature. Nevertheless, in spite of these unavoidable obstacles, Stockman and Kramer[1] provided numerical solutions for the temperature field of rectangular radiating fins whose properties vary linearly with temperature. Hung and Appl[2] developed a procedure which yields bounding functions for the temperature distribution of convective-radiative fins of arbitrary profiles. They considered properties that are linear functions of temperature. Shouman[3] derived an expression for the temperature variation of the previous problem extended to the case where the temperature-dependent properties are nonlinear.

Briefly, the present investigation is concerned with the conduction-radiation process in rectilinear fins of constant cross section dissipating heat to the surroundings at constant equivalent temperature. The fin material possesses properties that vary with temperature in a power fashion. This model corresponds to the fitting of piecewise continuous functions of the property-temperature curves.[4] The prime objective of this study is to illustrate a mathematical scheme for obtaining a solution for the radiative heat transfer along the aforesaid fins. This can be achieved by means of an algebraic transformation. Essentially, its aim is to convert the nonlinear differential equation to another of simpler form. Even though the resulting equation is still nonlinear, it may be manageable by available methods in a much easier way. In order to exemplify this, the case where the equivalent environment temperature becomes zero is analyzed. It presents an interesting and peculiar feature inasmuch as the form of the transformed differential equation is similar to the equation that results when the properties are constant. Therefore, the transformation is coupled with the results presented by Liu,[5] who solved the fin problem where the properties are invariant with temperature.

Formulation of the Problem

Consideration is focused on a rectangular thin fin of finite length L and thickness b. Heat is rejected by radiation to an environment whose equivalent temperature is T_s. It is assumed that the radiant interchange between the fin and its base is negligible. Accordingly, the temperature field $T(x)$ at any point along the fin satisfies the energy equation

$$(d/dx)(k\, dT/dx) - (2\varepsilon\sigma/b)(T^4 - T_s^4) = 0 \qquad (1)$$

in addition to the boundary conditions

$$T(0) = T_b \qquad (2)$$

$$dT(L)/dx = 0 \qquad (3)$$

The temperature variation of the thermal properties is represented by means of the relations

$$k = k_0 T^n \qquad (4)$$

$$\varepsilon = \varepsilon_0 T^a \qquad (5)$$

in the temperature interval under consideration $[T_L, T_b]$.

Substituting Eqs. (4) and (5) into Eq. (1) gives

$$(d^2T/dx^2) + (n/T)(dT/dx)^2 - (2\varepsilon_0\sigma/k_0b)T^{a-n}(T^4 - T_s^4) = 0 \tag{6}$$

whose main impediment, due to its inherent nonlinearity, can be circumvented by introduction of the algebraic transformation[6]

$$T = \theta^{1/(n+1)} \qquad n \neq -1 \tag{7}$$

Accordingly, the system is recast as follows

$$(d^2\theta/dx^2) - \beta\theta^{a/(n+1)}[\theta^{4/(n+1)} - \theta_s^{4/(n+1)}] = 0 \tag{8}$$

$$\theta(0) = \theta_b \tag{9}$$

$$d\theta(L)/dx = 0 \tag{10}$$

where $\beta = 2(n+1)\varepsilon_0\sigma/k_0b$.

Clearly, Eq. (8) is more amenable to mathematical treatment than the original Eq. (6). Hence, the task of obtaining its solution by any suitable method has been greatly alleviated.

Temperature distribution

For situations where the equivalent environment temperature is zero, Eq. (8) becomes

$$(d^2\theta/dx^2) - \beta\theta^\lambda = 0 \tag{11}$$

where $\lambda = (a+4)/(n+1)$. Moreover, it is desirable to include an additional condition, i.e.,

$$\theta(L) = \theta_L \tag{12}$$

which will be used as a parameter and evaluated later.

At this point, it must be recognized that the form of Eq. (11) is exactly identical to the prevailing energy equation when property variations are ignored.

For convenience, the dimensionless variables

$$\xi = x/L \qquad \psi = \theta/\theta_b \tag{13}$$

are adopted permitting Eq. (11) to be expressed as follows

$$(d^2\psi/d\xi^2) - N\psi^\lambda = 0 \tag{14}$$

where $N = \beta L^2\theta_b^{(\lambda-1)}$ is a modified radiation parameter. The pertinent boundary conditions in terms of ξ and ψ become

$$\psi(0) = 1 \tag{15}$$

$$d\psi(1)/d\xi = 0 \tag{16}$$

$$\psi(1) = \psi_r \tag{17}$$

Examination of the system composed of Eqs. (14–17) shows its analogy with that solved by Liu[5]. Using his technique, the exact temperature distribution of the fin can be written in terms of the complete and incomplete Beta functions as follows

$$B(\nu, \tfrac{1}{2}) - B_{(T_L/T)^{a+n+5}}(\nu, \tfrac{1}{2}) = \Gamma(T_L/T_b)^{(a-n+3)/2}[1 - x/L] \tag{18}$$

where $\nu = (\lambda - 1)/2(\lambda + 1)$ and $\Gamma = [2(\lambda+1)N]^{1/2}$.

The parameter T_L in Eq. (18) still remains to be evaluated. This can be accomplished by solving the equation which results from combining Eqs. (18) and (2); i.e.,

$$B(\nu, \tfrac{1}{2}) - B_{(T_L/T_b)^{a+n+5}}(\nu, \tfrac{1}{2}) = \Gamma(T_L/T_b)^{(a-n+3)/2} \tag{19}$$

Fin effectiveness

The heat-transfer rate per unit width from the fin, in terms of the variables ξ and ψ, may be evaluated as follows

$$q = -\frac{bk_0\theta_b}{(n+1)L}\frac{d\psi}{d\xi}\bigg|_{\xi=0} \tag{20}$$

Alternatively, introducing Eq. (7) into Eq. (20) and rearranging yields

$$q = \left(\frac{4\varepsilon_0\sigma k_0 b}{a+n+5}\right)^{1/2}(T_b^{a+n+5} - T_L^{a+n+5})^{1/2} \tag{21}$$

Finally, the resulting expression for the fin effectiveness, defined as the ratio of the actual heat loss to that of the ideal heat loss from a fin having a temperature T_b throughout, can be written as follows

$$\eta = \frac{1}{LT_b^{a+4}}\left[\frac{k_0b}{\varepsilon_0\sigma(a+n+5)}\right]^{1/2}[T_b^{a+n+5} - T_L^{a+n+5}]^{1/2} \tag{22}$$

Fin optimization

Attention is now turned to the optimization of the fin whose profile area ($A = bL$) is a specified quantity. The essential idea is to determine the optimum fin profile, so that the heat rejection attains a maximum. Inspection of Eq. (21) reveals that q depends on b and T_L. Thus, calculation of the maximum value of q, i.e.,

$$q(b, T_L) = \left(\frac{4\varepsilon_0\sigma k_0}{a+n+5}\right)^{1/2} b^{1/2}(T_b^{a+n+5} - T_L^{a+n+5})^{1/2} \tag{21}$$

subject to the constraint condition Eq. (19), rewritten as

$$g(b, T_L) = B(\nu, \tfrac{1}{2}) - B_{(T_L/T_b)^{a+n+5}}(\nu, \tfrac{1}{2}) - Mb^{-3/2}T_L^{(a-n+3)/2} = 0 \tag{23}$$

in which $M = [4(a+n+5)\varepsilon_0\sigma A^2/k_0]^{1/2}$, enables the obtainment of the corresponding optimum dimensions. Following Ref. 5, the optimization problem is solved by the method of Lagrange multipliers. Consequently, after some manipulations, the result is

$$b^{1/2} = -\frac{R[T_b^{a+n+5} - (1+S)T_L^{a+n+5}]^{1/3}}{(T_b^{a+n+5} - T_L^{a+n+5})^{1/6}} \tag{24}$$

in which

$$R = [\varepsilon_0\sigma A^2(a+n+3)^2/k_0(a+n+5)T_b^{2(n+1)}]^{1/6}$$

and

$$S = 3(a+n+5)/(a+n+3)$$

The combination of Eqs. (23) and (24) serves to determine the optimum fin thickness designated by b_{opt}. This is possible when T_L is eliminated in favor of b.

Ultimately, upon comparing Eqs. (21) and (24), it is readily shown that $(T_L/T_b)^{a+n+5} > (1+S)^{-1}$.

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